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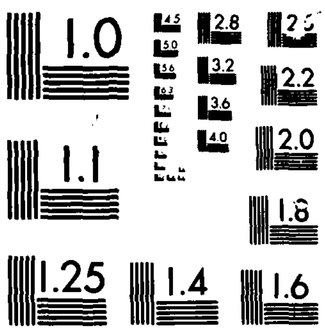

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ERGODICITY AND MIXING IN A FIVE-MODE SPECTRAL MODEL OF TWO-DIMENSIONAL TURBULENCE

by

M. Calvin Mosher

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COMPUTATION, MATHEMATICS, AND LOGISTICS DEPARTMENT
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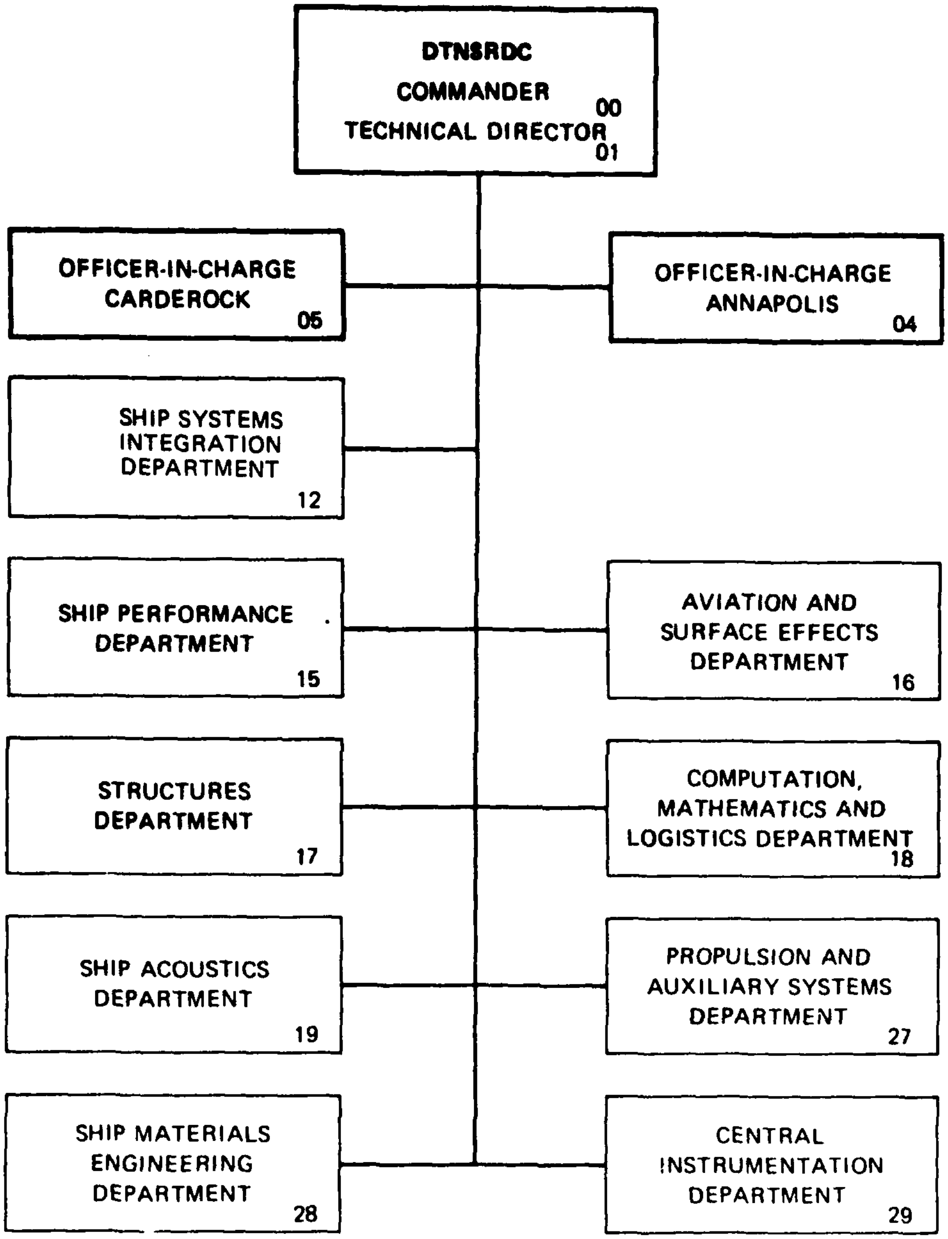

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A dynamical system derived by truncating to five frequencies a Fourier representation of the Euler equations for incompressible, two-dimensional fluid flow is investigated. In using Fourier representations to model turbulence, investigators have assumed that the truncated systems are ergodic and mixing, although the validity of this assumption is an open question. The truncated Fourier system has seven independent constants of motion which define a three-dimensional torus. Since a component of the solution to the system is quasi-periodic, the flow of the system can never be mixing and, in particular, cannot be mixing on the three-dimensional torus defined by the constants of motion.

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ABSTRACT

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INTRODUCTION

Truncated spectral representations of the Euler equations have been used in many theoretical and numerical investigations of inviscid, incompressible, two-dimensional fluid flow, especially, turbulent flow.[1#] The representations are measure-preserving dynamical systems which conserve energy and total vorticity or enstrophy. In analogy with statistical mechanics, the Fourier models are assumed to be ergodic[##] and mixing[4] on the set defined by the conserved energy and enstrophy, although the validity of this assumption is an open question. We will investigate a truncated system of five frequencies investigated by Kraichnan,[5] Hald,[6] and Glaz[7] and, in particular, the question of whether the system is mixing.

The spectral models are derived by representing the vorticity of an inviscid, incompressible, two-dimensional flow field as a Fourier series with complex coefficients ω_k, plugging this representation into the vorticity equation, and truncating

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[#]A complete listing of references is given on page 21.

[##]Definitions of ergodicity, mixing, invariant set, and quasi-periodic can be found in References 2 and 3.

the resulting infinite system of ordinary differential equations after a finite number of frequencies. For each frequency k the resulting equation[8] is

$$d\omega_k/dt = 0.5 \sum_{p+q=k} (q^{-2} - p^{-2})|p,q|\omega_p\omega_q$$

where $p = (p_1,p_2)$, $q = (q_1,q_2)$, $p^2 = p_1^2+p_2^2$, $|p,q| = p_1q_2-p_2q_1$, and d/dt is the derivative with respect to time t. It is assumed that ω_{-k} is the complex conjugate of ω_k, i.e., $\omega_{-k} = \omega_k^*$. The summation $p+q = k$ is extended over only a finite number of frequencies. These systems have at least two constants of motion, namely, the energy E and the squared vorticity or enstrophy Ω

$$E = 0.5 \sum_p p^{-2}\omega_p\omega_p^*, \quad \Omega = \sum_p \omega_p\omega_p^*$$

Here the summation is taken over all the frequencies in the truncation. Using a truncation which retains the five frequencies $k_1 = (1,1)$, $k_2 = (2,1)$, $k_3 = (3,0)$, $k_4 = (2,-1)$, and $k_5 = (1,-1)$, we obtain the system

$$d\omega_1/dt = 8\omega_3\omega_4^* \tag{1a}$$

$$d\omega_2/dt = 35\omega_3\omega_5^* \tag{1b}$$

$$d\omega_3/dt = 27(\omega_1\omega_4 - \omega_2\omega_5) \tag{1c}$$

$$d\omega_4/dt = -35\omega_3\omega_1^* \tag{1d}$$

$$d\omega_5/dt = -8\omega_3\omega_2^* \tag{1e}$$

The system has these constants of motion:[6]

$$\gamma_1 = 35r_1^2 + 8r_4^2 \tag{2a}$$

$$\gamma_2 = 35r_5^2 + 8r_2^2 \tag{2b}$$

$$\gamma_3 = 35r_3^2 + 27r_2^2 + 27r_4^2 \tag{2c}$$

$$I_1 = \omega_1\omega_2^* + \omega_4^*\omega_5 \tag{2d}$$

$$I_2 = 35\omega_1\omega_5^* - 8\omega_2\omega_4^* \tag{2e}$$

$$I_3 = \mathrm{Im}(\omega_1\omega_3^*\omega_4 + \omega_2^*\omega_3\omega_5^*) \tag{2f}$$

Here $r_i = |\omega_i|$ for $i=1,\ldots,5$. The energy $E = (\gamma_1+\gamma_2)/140 + \gamma_3/630$ and the enstrophy $\Omega = (\gamma_1+\gamma_2+\gamma_3)/35$. However, γ_1, γ_2, γ_3, I_1, I_2, and I_3 have no clear physical interpretation.

Hald[6] discovered the constants of motion of Equations (2). Since the system has extra constants of motion in addition to E and Ω, the flow of the system cannot be ergodic or mixing on the set defined by E and Ω. However, the question remains of whether the system is ergodic or mixing on the set S defined by the new constants of motion, i.e., the set of points ω_1, ω_2, ω_3, ω_4, ω_5 that satisfy Equations (2). Also, the topology of S is of interest.

We have obtained the following results. The eight constants of motion discovered by Hald are not independent. A parameterization of the set S defined by the remaining seven constants of motion shows that S is a three-dimensional torus. That S is a three-dimensional manifold implies the seven constants of motion are independent. The ω_3 component of the solution to the system can be solved explicitly using a Jacobian elliptic function. Since the ω_3 component is quasi-periodic, the flow of the system can never be mixing and, in particular, cannot be mixing on S. The results of a numerical test are consistent with the parameterization of S.

In the following sections of this report we will first paramterize the set S and then derive an explicit solution of the $\omega_3(t)$ component of Equations (1). Mixing will be discussed next. Finally, the results of the numerical test will be described.

PARAMETERIZATION OF THE INVARIANT SET S

The parameterization of the set S defined by the constants of motion is similar to that of the four-mode case.[9] Letting $\omega_j = r_j e^{i\theta_j}$ for $j=1,\ldots,5$ and $i = \sqrt{-1}$, we will show that the projection of S onto r_2, r_4 space is a set R homeomorphic to a rectangle. The variables r_1, r_3, r_5, θ_1, θ_3, θ_4, and θ_5 can be parameterized by θ_2 and the r_2, r_4 which lie in R. We will first determine the set R' which will be the projection of the set defined by Equations (2a) through (2e) onto r_2, r_4 space. We will find that these equations are not independent, but the following relation holds

$$\gamma_1\gamma_2 - 280|I_1|^2 - |I_2|^2 = 0$$

Therefore, only seven of the eight Equations (2a) through (2f) can be independent. Next, we will show that Equation (2f), which is used to parameterize θ_3, defines an annulus. The intersection of R' and the annulus will be R.

Letting $s = r_2$ and $t = r_4$, we rewrite Equations (2d) and (2e) as follows:

$$-1 + r_1 s|I_1|^{-1} e^{i(\theta_1-\theta_2-\psi_1)} = tr_5|I_1|^{-1} e^{i(\theta_5-\theta_4-\psi_1+\pi)} \tag{3a}$$

$$-1 + 35r_1r_5|I_2|^{-1} e^{i(\theta_1-\theta_5-\psi_2)} = 8st|I_2|^{-1} e^{i(\theta_2-\theta_4-\psi_2)} \tag{3b}$$

We can interpret these equations by using triangles in the complex plane with $i = \sqrt{-1}$. Let $z_1 = x_1 + iy_1$ be the value of the right-hand side of Equation (3a). Then, s, t, r_1, and r_5 satisfy Equation (3a) if and only if the triangle ABC in Figure 1 exists. We will call this the first triangle. Here AB is the complex number $r_1 s|I_1|^{-1} e^{i(\theta_1-\theta_2-\psi_1)}$, CB is $tr_5|I_1|^{-1} e^{i(\theta_5-\theta_4-\psi_1+\pi)}$, the angle CAB = $\arg(z_1+1) = \theta_1-\theta_2-\psi_1$, and $\arg(z_1) = \theta_5-\theta_4-\psi_1+\pi$. Similarly, let $z_2 = x_2+iy_2$ be the value of the right-hand side of Equation (3b). Then, s, t, r_1, and r_5 satisfy Equation (3b) if and only if the triangle DEF in Figure 2 exists. We will call this the second triangle. Here DE is the complex number $35r_1r_5|I_2|^{-1} e^{i(\theta_1-\theta_5-\psi_2)}$, FE is $8st|I_2|^{-1} e^{i(\theta_2-\theta_4-\psi_2)}$, the angle FDE = $\arg(z_2+1) = \theta_1-\theta_5-\psi_2$, and $\arg(z_2) = \theta_2-\theta_4-\psi_2$.

The equations for the angles obtained from the triangle representations of Equations (3), i.e.,

$$\theta_1 - \theta_2 = \arg(z_1+1) + \psi_1 \tag{4a}$$

$$\theta_5 - \theta_4 = \arg(z_1) + \psi_1 - \pi \tag{4b}$$

$$\theta_1 - \theta_5 = \arg(z_2+1) + \psi_2 \tag{4c}$$

$$\theta_2 - \theta_4 = \arg(z_2) + \psi_2 \tag{4d}$$

can be reduced to the following four equations:

$$\theta_5 = \theta_2 + \arg(z_1+1) - \arg(z_2+1) + \psi_1 - \psi_2 \tag{5a}$$

$$\theta_4 = \theta_2 - \arg(z_2) - \psi_2 \tag{5b}$$

$$\theta_1 = \theta_2 + \arg(z_1+1) + \psi_1 \tag{5c}$$

$$\arg(z_1) - \arg(z_1+1) + \arg(z_2+1) - \arg(z_2) = \pi \tag{5d}$$

Hence, angles θ_1, θ_4, and θ_5 can be parameterized by θ_2, z_1, and z_2. We will show that z_1 and z_2 can in turn be parameterized by s and t. First, however, we need to examine Equation (5d).

As seen in Figures 1 and 2,

$$\arg(z_1) = \arg(z_1+1) + \text{angle ABC} \tag{6a}$$

and

$$\arg(z_2) = \arg(z_2+1) + \text{angle DEF} \tag{6b}$$

Using Equations (6), we convert Equation (5d) to the form

$$\cos(\text{angle ABC}) + \cos(\text{angle DEF}) = 0 \tag{7}$$

Using the law of cosines on the triangles in Figures 1 and 2, we obtain

$$\cos(\text{angle ABC}) = (r_1^2\, s^2 + r_5^2\, t^2 - |I_1|^2)/(2r_1 r_5 st) \tag{8a}$$

$$\cos(\text{angle DEF}) = (35^2\, r_1^2\, r_5^2 + 64 s^2\, t^2 - |I_2|^2)/(560 r_1 r_5 st) \tag{8b}$$

Using Equations (8) with r_1 and r_5 expressed in terms of s and t by Equations (2a) and (2b), we transform Equation (7) to the form

$$\gamma_1\gamma_2 - 280|I_1|^2 - |I_2|^2 = 0 \tag{9}$$

Hence, only seven of the eight Equations (2a) through (2f) can be independent. Equation (9) can in turn be transformed into the following three equivalent equations which will be useful in parameterizing z_1 and z_2 in terms of s and t:

$$-8\gamma_2 + 4|I_2|^{-2}(\gamma_1\gamma_2 + |I_2|^2)\gamma_2 = 1120\gamma_2|I_1|^2|I_2|^{-2} \tag{10a}$$

$$-8\gamma_1 + 4|I_2|^{-2}(\gamma_1\gamma_2 + |I_2|^2)\gamma_1 = 1120\gamma_1|I_1|^2|I_2|^{-2} \tag{10b}$$

$$\gamma_1\gamma_2 - |I_2|^{-2}(\gamma_1\gamma_2 + |I_2|^2)^2/4 = -140^2|I_1|^4|I_2|^{-2} \tag{10c}$$

To parameterize z_1 and z_2 in terms of s and t, we use the following equations obtained from the triangle representations of Equations (2d) and (2e) shown in Figures 1 and 2:

$$x_1^2 + y_1^2 = |I_1|^{-2}\, t^2\, r_5^2 \tag{11a}$$

$$(x_1+1)^2 + y_1^2 = |I_1|^{-2}\, r_1^2\, s^2 \tag{11b}$$

$$x_2^2 + y_2^2 = 64|I_2|^{-2}\, s^2\, t^2 \tag{11c}$$

$$(x_2+1)^2 + y_2^2 = 35^2 \, |I_2|^{-2} \, r_1^2 \, r_5^2 \tag{11d}$$

Expressing r_1 and r_5 in terms of s and t by Equations (2a) and (2b) and using Equations (9) and (10), we can solve Equations (11) for x_1, x_2, y_1^2, and y_2^2 in terms of s and t:

$$x_1 = |I_1|^{-2}(\gamma_1 s^2 - \gamma_2 t^2 - 35|I_1|^2)/70 \tag{12a}$$

$$x_2 = -4|I_2|^{-2}(\gamma_1 s^2 + \gamma_2 t^2 - 35|I_1|^2) \tag{12b}$$

$$y_1^2 = -70^{-2}|I_1|^{-4}(\gamma_1 s^2 + \gamma_2 t^2 + 2|I_2|st-35|I_1|^2)\cdot$$

$$\cdot(\gamma_1 s^2 + \gamma_2 t^2 - 2|I_2|st-35|I_1|^2) \tag{12c}$$

$$y_2^2 = 16|I_2|^{-4}70^2|I_1|^4 y_1^2 \tag{12d}$$

We see that the equations $y_2^2 = 0$ and $y_1^2 = 0$ are identical. The expression within each pair of parentheses of Equation (12c), when equated to zero, represents an ellipse. The length of the minor and major axes of the ellipses are, respectively,

$$\beta_- = 2(70)^{1/2} \, |I_1|/[\gamma_1+\gamma_2 + ((\gamma_1+\gamma_2)^2 - 1120|I_1|^2)^{1/2}]^{1/2} \tag{13a}$$

$$\beta_+ = 2(70)^{1/2} \, |I_1|/[\gamma_1+\gamma_2 - ((\gamma_1+\gamma_2)^2 - 1120|I_1|^2)^{1/2}]^{1/2} \tag{13b}$$

The graph of the equation $y_1^2 = 0$ is shown in Figure 3. R' is the set of (s,t) for which the first and second triangles exist, or, equivalently, the set for which the right-hand side of Equation (12c) is non-negative. Since the right-hand side of Equation (12c) is negative at the origin and at infinity and s,t are non-negative, R' must be the set of points lying within one ellipse but in the complement of the other in the first quadrant of the s,t plane as seen in Figure 3.

At this point the angles θ_1, θ_4, and θ_5 have been parameterized by z_1, z_2, and θ_2 using Equations (5) which were derived from Equations (2d) and (2e). Since r_1, r_5, r_3, z_1, and z_2 were parameterized by the s,t in R' using Equations (2a), (2b),

(2c), (2d), and (2e), respectively, the set defined by Equations (2a) through (2e) is parameterized by s,t, and θ_2 with the (s,t) lying in R' and no constraint on the values of θ_2. The only remaining variable is θ_3 which will be parameterized by Equation (2f).

We will now determine the s,t which satisfy Equation (2f). Letting

$$A = r_1 t \sin(\theta_1+\theta_4) - s\, r_5 \sin(\theta_2+\theta_5) \tag{14a}$$

$$B = -r_1 t \cos(\theta_1+\theta_4) + s\, r_5 \cos(\theta_2+\theta_5) \tag{14b}$$

$$C = I_3/r_3 \tag{14c}$$

$$x = \cos(\theta_3) \tag{14d}$$

$$y = \sin(\theta_3)\ , \tag{14e}$$

we can write Equation (2f) as

$$C = Ax + By \tag{15}$$

Solving for θ_3, we have

$$\theta_3^+ = \arccos(\ (AC + [B^2(A^2 + B^2 - C^2)]^{1/2})/(A^2 + B^2)\ ) \tag{16a}$$

$$\theta_3^- = \arccos(\ (AC - [B^2(A^2 + B^2 - C^2)]^{1/2})/(A^2 + B^2)\ ) \tag{16b}$$

Since the arccos function is multivalued, it appears that θ_3 could have four solutions. However, for fixed values of A, B, and C, Equation (15) defines a line in the variables x and y. The simultaneous solutions in x and y to Equations (14d), (14e), and (15) will be the intersection points of the line and the circle $x^2 + y^2 = 1$. Hence, for fixed values of A, B, and C, Equations (14d), (14e), and (15) can be solved by at most two values of θ_3 which must be determined by Equations (16a) and (16b), respectively. Consequently, θ_3^+ and θ_3^- are single-valued. When the line is tangent to the circle, θ_3 has one solution and θ_3^+ equals θ_3^-,

i.e., θ_3 has two solutions if $A^2 + B^2 - C^2$ is positive and one solution if $A^2 + B^2 - C^2$ is zero.

Finding the s,t which satisfy Equation (2f) is now reduced to finding the s,t for which $A^2 + B^2 - C^2$ is non-negative. Defining $u = s^2 + t^2$ and $P(u) = A^2 + B^2 - C^2$, we have

$$P(u) = -8u^2/35 + (\gamma_1+\gamma_2)u/35 - |I_1|^2 - 35|I_3|^2/(\gamma_3-27u)$$

We will show that the values of u satisfying Equations (2) must lie between two roots of P(u) in the interval $[\beta_-^2, \beta_+^2]\cap(0,\gamma_3/27)$. Let $Q(u) = (\gamma_3-27u)P(u)$. Since Q(u) is a cubic polynomial in u, it has three roots. Consequently, P(u) also has three roots because $Q(\gamma_3/27) = -35|I_3|^2$ with I_3 being, in general, nonzero. Since P(u) is positive as u approaches $\gamma_3/27$ from the right but negative as u approaches $+\infty$, P(u) has one real root greater than $\gamma_3/27$.

However, the two remaining roots of P(u) must lie in the interval $[\beta_-^2, \beta_+^2]\cap(0,\gamma_3/27)$. In order that Equation (15) be solvable for nontrivial values of θ_3, P(u) must be positive for some value of u' corresponding to a point (s', t') in R'. In particular, u' must be less than $\gamma_3/27$ to satisfy Equation (2c). Hence, P(u) must be positive for some values of u in the interval $[\beta_-^2, \beta_+^2]\cap(0,\gamma_3/27)$. Since P(u) is negative at both β_-^2 and β_+^2 and is also negative as u approaches $\gamma_3/27$ from the left, P(u) must have its two remaining roots u_1, u_2 in $[\beta_-^2, \beta_+^2]\cap(0,\gamma_3/27)$. Hence, P(u) is non-negative for (s,t) in R' which lie in the annulus with inner radius $\sqrt{u_1}$ and outer radius $\sqrt{u_2}$ assuming u_1 is less than u_2.

We conclude that the s,t for which θ_3 is defined, i.e., the s,t for which $A^2 + B^2 - C^2$ is non-negative, lie in the annulus with radii $\sqrt{u_1}$ and $\sqrt{u_2}$. The set R shown in Figure 4 is the intersection of this annulus and R'. R is the set of points in s,t space which we use to parameterize the set S.

The coordinates θ_1, θ_3, θ_4, and θ_5 of the points in S have now been parameterized by the coordinates s,t, and θ_2 of points in the set $R \times [0,2\pi]$. Equations (5) are used to parameterize θ_1, θ_4, and θ_5 and Equations (16) are used for θ_3. Equations (2a), (2b), and (2c) are used to parameterize r_1, r_5, and r_3, respectively. Collecting these equations, we have:

$$r_1 = [(\gamma_1 - 8t^2)/35]^{1/2} \tag{17a}$$

$$r_2 = s \tag{17b}$$

$$r_3 = [(\gamma_3 - 27(s^2+t^2))/35]^{1/2} \tag{17c}$$

$$r_4 = t \tag{17d}$$

$$r_5 = [(\gamma_2 - 8t^2)/35]^{1/2} \tag{17e}$$

$$\theta_1 = \theta_2 + \arg(z_1+1) + \psi_1 \tag{17f}$$

$$\theta_2 = \theta_2 \tag{17g}$$

$$\theta_4 = \theta_2 - \arg(z_2) - \psi_2 \tag{17h}$$

$$\theta_5 = \theta_2 + \arg(z_1+1) - \arg(z_2+1) + \psi_1 - \psi_2 \tag{17i}$$

$$\theta_3^+ = \begin{cases} \operatorname{arccos}_1(\Phi^+) & \text{if } \lambda(\operatorname{arccos}_1(\Phi^+)) = 0 \quad (17j) \\ \operatorname{arccos}_2(\Phi^+) & \text{if } \lambda(\operatorname{arccos}_2(\Phi^+)) = 0 \quad (17k) \end{cases}$$

$$\theta_3^- = \begin{cases} \operatorname{arccos}_1(\Phi^-) & \text{if } \lambda(\operatorname{arccos}_1(\Phi^-)) = 0 \quad (17l) \\ \operatorname{arccos}_2(\Phi^-) & \text{if } \lambda(\operatorname{arccos}_2(\Phi^-)) = 0 \quad (17m) \end{cases}$$

$$\Phi^+ = (AC + [B^2(A^2 + B^2 - C^2)]^{1/2})/(A^2 + B^2)$$

$$\Phi^- = (AC - [B^2(A^2 + B^2 - C^2)]^{1/2})/(A^2 + B^2)$$

$$\lambda(\Theta) = C - A\cos(\Theta) - B\sin(\Theta)$$

Here $\lambda(\Theta) = 0$ is Equation (2f) and arccos_i for $i = 1,2$ are the two branches of the arccos function taking values between 0 and π and between π and 2π, respectively.

The variables A, B, and C are defined by Equations (14a), (14b), and (14c), respectively, and z_1 and z_2 are defined by Equations (12).

Equations (17), however, cannot be used in their present form as a one-to-one parameterization of S because θ_3 has multiple values. In addition, the signs of y_1 and y_2 are undetermined implying both branches of the arg function must be used in Equations (17f), (17h), and (17i). As indicated in the derivation of Equations (16), θ_3 has two values for any $< s, t, \theta_2 >$ in $R \times [0, 2\pi]$, i.e., θ_3^+ and θ_3^-. Also, for any $< s, t >$ in R Equations (17) do not specify the signs of $y_1(s,t)$ and $y_2(s,t)$ implying $\arg(z_i)$ and $\arg(z_i+1)$ for $i = 1, 2$ will each have two values for any $< s, t >$ in R. However, only the sign of y_1 needs to be determined because we can show that y_1 and y_2 have opposite signs for any s,t in R. The argument proceeds as follows. Since $\arg(z_1) = \arg(z_1+1)$ + angle ABC as seen in Figure 1, we have

$$\arg(z_1) - \arg(z_1+1) \geq 0 \quad \text{for } y_1 \geq 0$$

and

$$\arg(z_1) - \arg(z_1+1) \leq 0 \quad \text{for } y_1 \leq 0$$

Exactly the same inequalities hold with z_1, y_1, and angle ABC replaced with z_2, y_2, and angle DEF, respectively, in Figure 2. If $y_1 \geq 0$, the angle ABC lies between 0 and π implying by Equation (5d) that

$$\arg(z_2) - \arg(z_2+1) = \text{angle DEF} \leq 0$$

and, therefore, $y_2 \leq 0$. Similarly, if $y_1 \leq 0$, then $y_2 \geq 0$.

To obtain a one-to-one parameterization of S, we can account for the two values of θ_3 and the sign of y_1 by reformulating Equations (17) as four maps $\psi_1, \ldots, \psi_4$ corresponding to four cases:

$$\theta_3 = \theta_3^+, \; y_1 \geq 0 \qquad \text{for case 1} \tag{18a}$$

$$\theta_3 = \theta_3^+, \; y_1 \leq 0 \qquad \text{for case 2} \tag{18b}$$

$$\theta_3 = \theta_3^-, \; y_1 \geq 0 \qquad \text{for case 3} \tag{18c}$$

$$\theta_3 = \theta_3^-, \; y_1 \leq 0 \qquad \text{for case 4} \tag{18d}$$

Each ψ_i is a one-to-one map from $R \times [0,2\pi]$ to S

$$\psi_i : R \times [0, 2\pi] \dashrightarrow S$$

for i = 1,...,4. The map ψ_1 is defined by Equations (17) with $\theta_3 = \theta_3^+$ and with the arg function in Equations (17f), (17h), and (17i) taking values between 0 and π, i.e., $y_1 \geq 0$. The map ψ_2 will also have values θ_3 defined by θ_3^+ but the arg function will take values between π and 2π, i.e., $y_1 \leq 0$. However, ψ_3 will have values θ_3 defined by θ_3^- and the arg function defined for $y_1 \geq 0$. The map ψ_4 will have values θ_3 defined by θ_3^- and the arg function defined for $y_1 \leq 0$.

To construct the parameterization ψ of S from ψ_1, ..., ψ_4 we need to paste together four copies of $R \times [0,2\pi]$. To facilitate the construction of ψ, we can view $R \times [0,2\pi]$ as a box because R is homeomorphic to a square. That is, R can be mapped by a homeomorphism onto a square R_{sq} in a space with new coordinates s',t' as shown in Figure 5; the two sides of R determined by the ellipses in Figure 4 are mapped to opposite sides of the square and the two sides determined by the annulus are mapped to the other two sides of the square. The four copies of $R_{sq} \times [0,2\pi]$ viewed as boxes can be pasted together with corresponding sides matched as shown in Figure 6 to form a composite box with periodic sides, i.e., a three-dimensional torus. Each map ψ_i will be defined on the i^{th} box in Figure 6 for i=1,...,4. On the surfaces of the boxes the ψ_i for i=1,...,4 have values in common because $\theta_3^+ = \theta_3^-$ for any < s, t, θ_2 > with < s, t > on the circles defined by $s^2 + t^2 = u_1$ and $s^2 + t^2 = u_2$ in Figure 4. Also, the two branches of the arg function (for $y_1 \geq 0$ and $y_2 \leq 0$) agree for any < s, t, θ_2 > with < s, t > on the ellipses defined by $y_1^2 = 0$ in Figure 4. Since each ψ_i is one-to-one and continuous on the i^{th} box and the ψ_i have values in common only on the surfaces of the boxes for i=1,...,4, we conclude that the ψ_1, ..., ψ_4 define a homeomorphism ψ from the three-dimensional torus consisting of four copies of $R_{sq} \times [0,2\pi]$ pasted together to the set S defined by the constants of motion.

EXPLICIT SOLUTION OF $\omega_3(t)$

An explicit solution of the $\omega_3(t)$ component of Equations (1) will be derived. We start by differentiating Equation (1c) with respect to time t. Using Equations (2a), (2b), and (2c) we obtain

$$d^2\omega_3/dt^2 = (\alpha+\beta r_3^2)\omega_3 \tag{19}$$

where $\alpha = 16\gamma_3 - 27(\gamma_1+\gamma_2)$ and $\beta = -560$. Converted to polar coordinates, Equation (19) can be represented as two equations

$$d^2r_3/dt^2 - \beta r_3^3 - \alpha r_3 - r_3(d\theta_3/dt)^2 = 0 \tag{20a}$$

$$2(dr_3/dt)\ (d\theta_3/dt) + r_3\ (d^2\theta_3/dt^2) = 0 \tag{20b}$$

Integrating the second equation with respect to t, we obtain

$$d\theta_3/dt = r_3^{-2}\ e^{c_1} \tag{21}$$

where c_1 is a constant. If we substitute $d\theta_3/dt$ into the first equation, multiply by dr_3/dt, integrate and multiply the result by r_3^2, the first equation becomes

$$(dy/dt)^2 = 2\beta y^3 + 4\alpha y^2 - 8c_2y - 4e^{2c_1} \tag{22}$$

where $y = r_3^2$ and c_2 is a constant from the integration. Equation (22) can be written in the form

$$(dy/dt)^2 = h^2(y-\lambda_1)(y-\lambda_2)(y-\lambda_3) \tag{23}$$

where $h^2 = 2\beta$ and λ_1, λ_2, and λ_3 are the roots of the right-hand side. This is an elliptic equation. It has an explicit solution[10]

$$r_1^2 = y = \lambda_3 + (\lambda_2-\lambda_3)\mathrm{sn}^2(hM(t+t_0),k) \tag{24}$$

where $M^2 = 0.25(\lambda_1-\lambda_3)$, $k^2 = (\lambda_2-\lambda_3)/(\lambda_1-\lambda_3)$, and sn is a Jacobian elliptic function. Hence, r_1 is periodic with period $\rho = 2K/hM$, where

$$K = \int_0^1 [(1-x^2)(1-k^2x^2)]^{-1/2}\, dx$$

Now $\theta_3(t)$ can be obtained by integrating Equation (21) with respect to t using Equation (24).

MIXING

We will show that the real and imaginary components of $\omega_3(t)$ are quasi-periodic. Letting x_3 and y_3 be the real and imaginary parts of ω_3, respectively, we can convert Equation (19) into two second order equations

$$d^2x_3/dt^2 = (\alpha+\beta r_3^2)x_3 \tag{25a}$$

$$d^2y_3/dt^2 = (\alpha+\beta r_3^2)y_3 \tag{25b}$$

Letting $v_1 = x_3$ and $v_2 = dx_3/dt$, we now convert (25a) into a first-order system of two equations

$$d\bar{v}/dt = H(t)\bar{v} \tag{26}$$

with $\bar{v} = (v_1, v_2)$ and H(t) a 2×2 matrix. Because $r_3^2(t)$ is periodic with period ρ, H(t) is periodic with the same period. By Floquet theory[11] there exists a periodic nonsingular matrix P(t) of period ρ and a constant matrix R such that a fundamental matrix $\Phi(t)$ of the system (26) can be represented as

$$\Phi(t) = P(t)e^{tR}$$

However, the eigenvalues of the matrix e^R must be pure imaginary because $r_3(t) = r_3(-t)$ making $x_3(t) = x_3(-t)$ by Equation (25a) and both x_3 and dx_3/dt are bounded as seen by inspection of Equations (1c), (2a), (2b), and (2c). Transforming e^R into Jordan canonical form and representing[12] the transformed matrix in exponential form $e^{R'}$, we see that the new fundamental matrix $\Phi'(t) = P'(t)e^{tR'}$ (P' being the transformed matrix P) must be quasi-periodic. Hence, $x_3(t)$ is quasi-periodic as is $y_3(t)$ by the same argument applied to Equation (25b).

Since $\omega_3(t)$ is quasi-periodic, the solution to Equations (1) can never, in general, be mixing. In particular, since $r_3(t)$ is periodic, the solution can never be mixing on the set S defined by the constants of motion.

NUMERICAL TEST

The Gear-Hindmarsh[13] ordinary differential equation solver was used to solve the system of Equations (1) for the initial point $\omega_1 = 1$, $\omega_2 = 1$, $\omega_3 = 1$, $\omega_4 = 1$, and $\omega_5 = \sqrt{-1}$. The parameter EPS was set at 10^{-7}. At every 10 time steps the solution was plugged into Equations (2) to ensure that the resulting constants of motion agreed to four digits with those of the initial point, namely, $\gamma_1 = 43$, $\gamma_2 = 43$, $\gamma_3 = 89$, $I_1 = 1 + \sqrt{-1}$, $I_2 = -8 - 35\sqrt{-1}$, and $I_3 = -1$. An Apollo DN420 computer was used with fifteen digits of accuracy (double precision).

When the system was integrated to $t = 1$, the trajectory of the system wound around the torus passing repeatedly through the four boxes in Figure 6. The trajectory projected onto s,t space intersects repeatedly the boundary of R but never crosses it as seen in Figure 7. That the trajectory appears in Figure 7 to be tangent to the boundary of R while never crossing it is evidence that the derivation of the region R is correct.

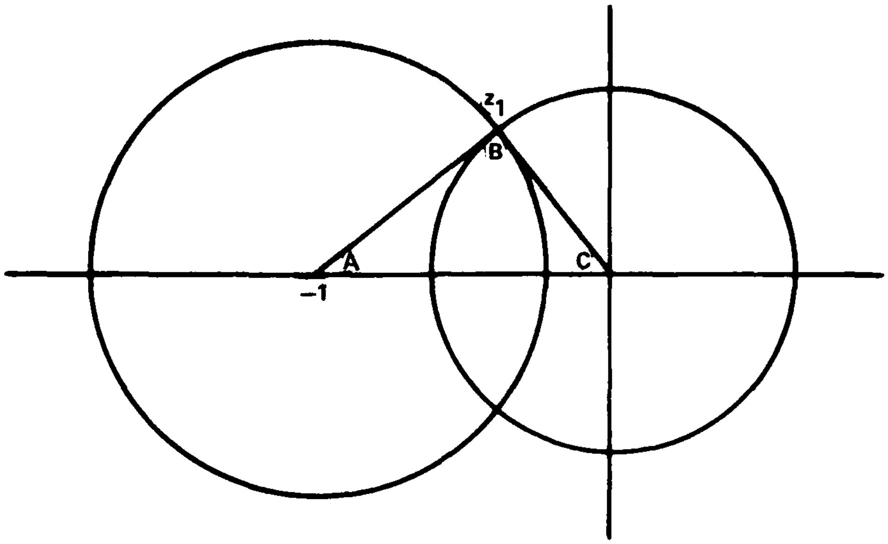

Figure 1 - First Triangle

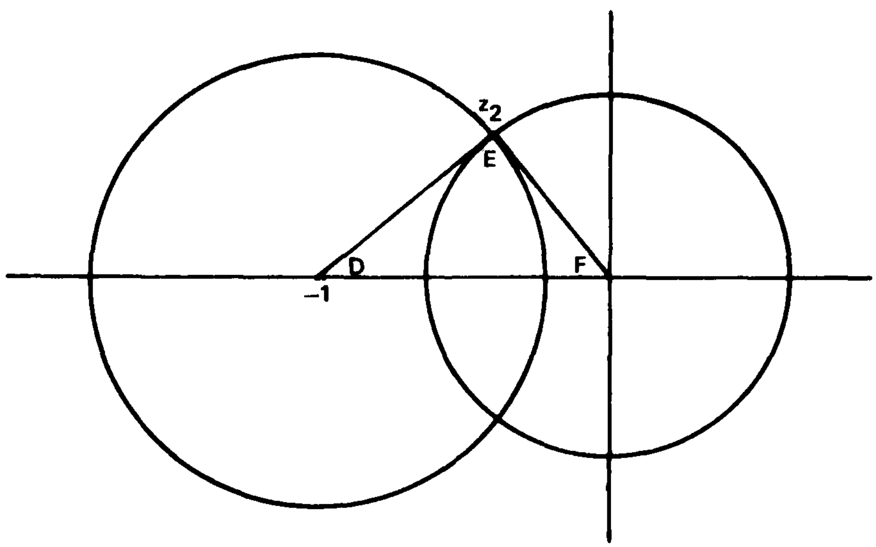

Figure 2 - Second Triangle

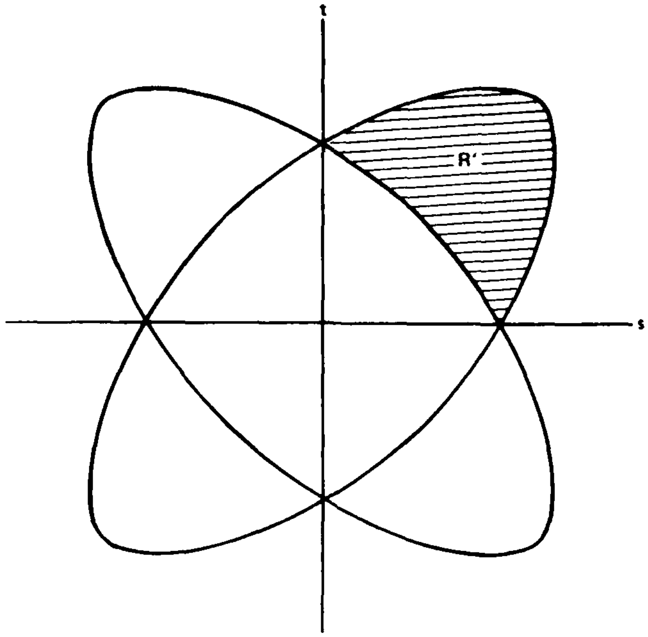

Figure 3 - Graph of $y_1^2 = 0$; R' is the Shaded Region

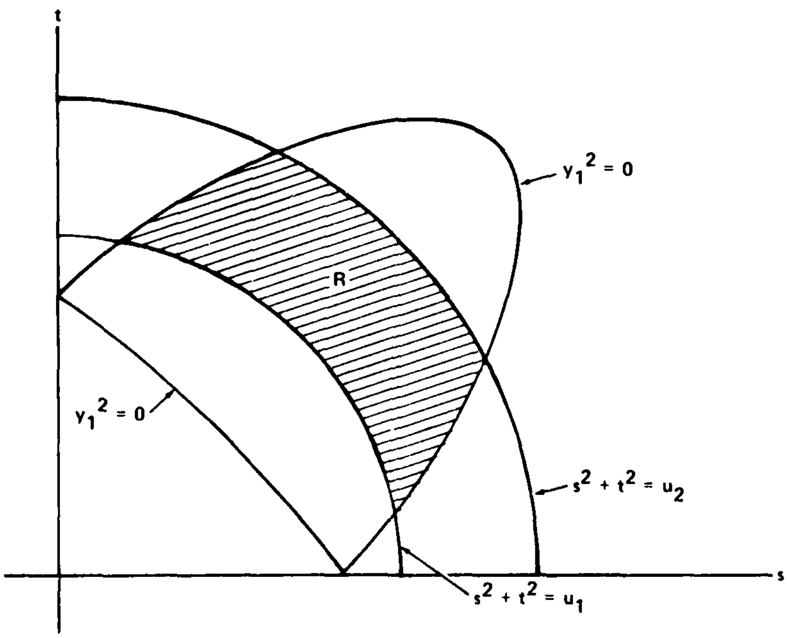

Figure 4 - The Set R Defined by the Intersection of R' and the Annulus with Inner Radius $\sqrt{u_1}$ and Outer Radius $\sqrt{u_2}$

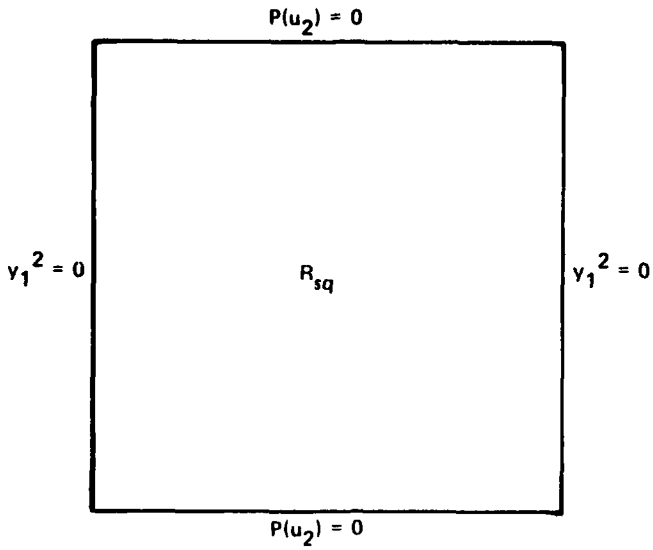

Figure 5 - The Square R_{sq} in s', t' Space Homeomorphic to R with Sides Labeled by Equations Defining Corresponding Sides of R

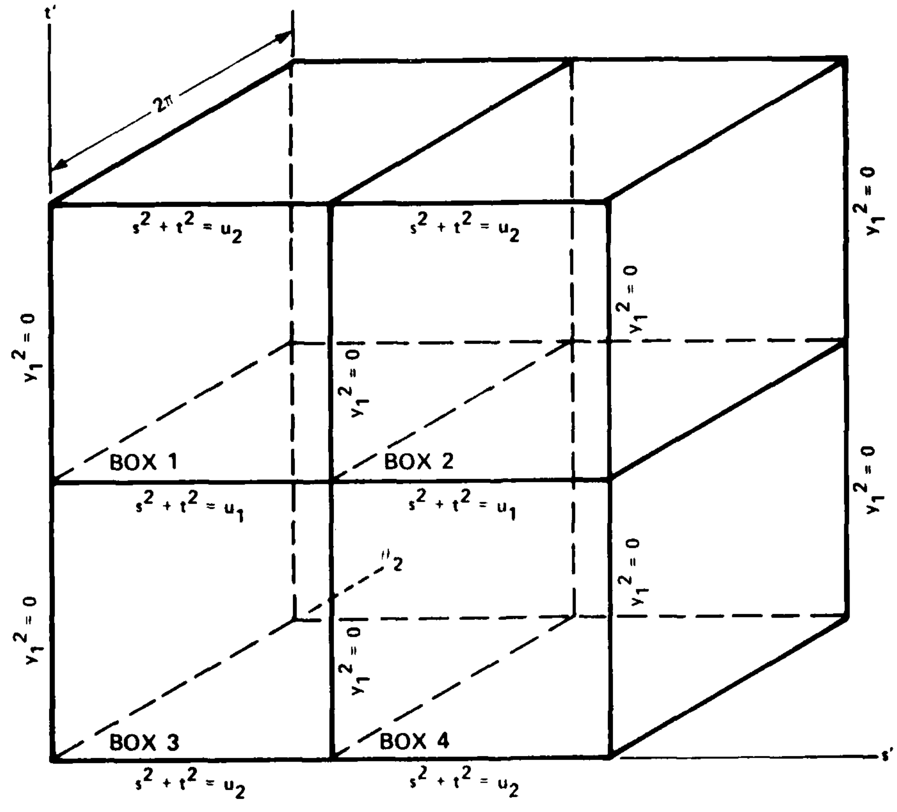

Figure 6 - Four Copies of the Box $R_{sq}\times[0,2\pi]$ Homeomorphic to a Three-Dimensional Torus

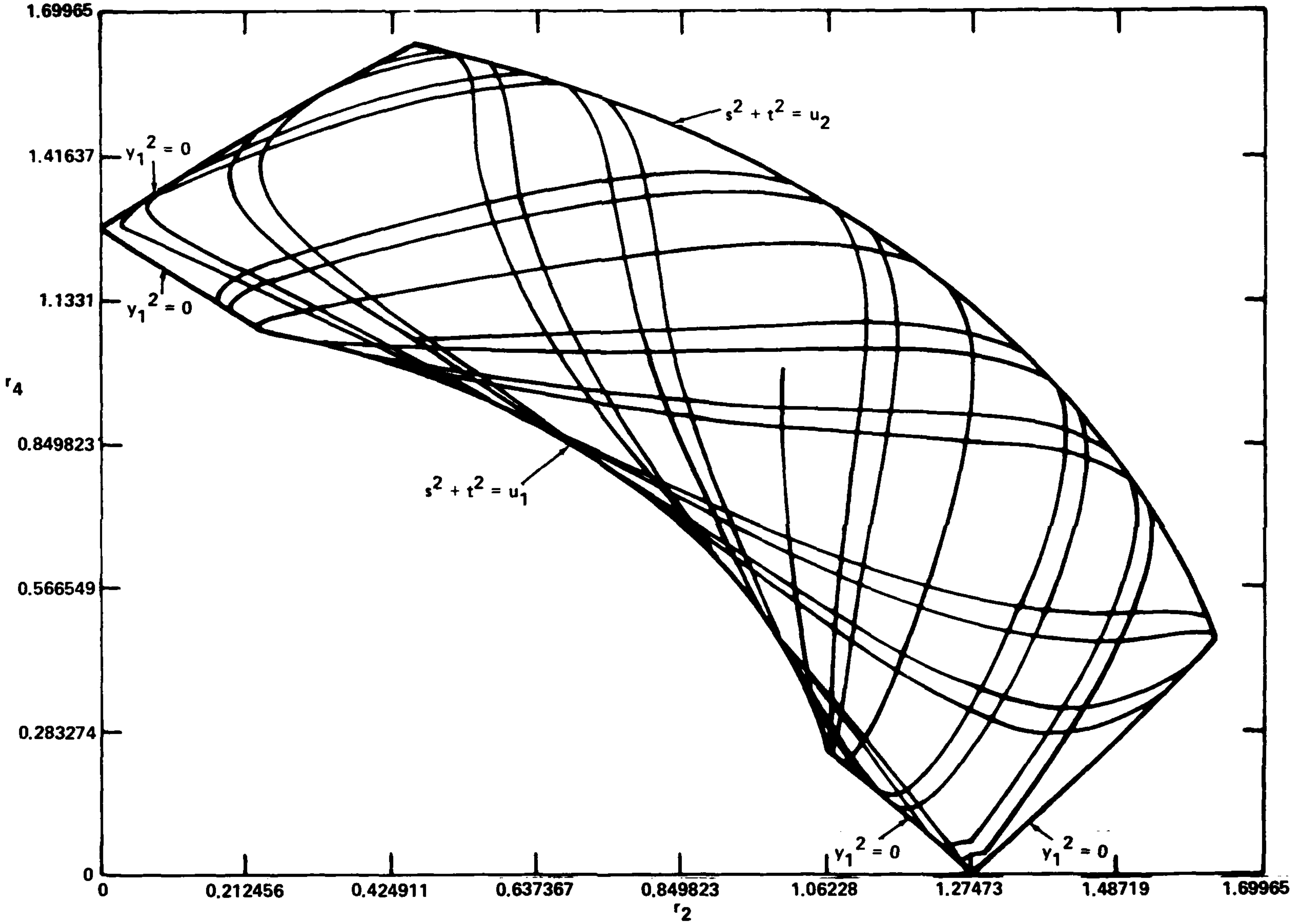

Figure 7 - Trajectory (———) of System Projected onto Region R with Boundary (———)

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